Evolution of Entropy with Cosmic Time

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Abstract

Understanding the evolution of entropy in the universe is a fundamental aspect of cosmology. This paper investigates the evolution of entropy in a spatially flat $K = 0$ universe, focusing on the contributions of matter, radiation, and dark energy components. The study derives the rate of change of entropy with respect to cosmic time, taking into account the scaling relations of energy densities and temperatures for different components. The analysis reveals the dominance of radiation entropy at early times, transitioning to matter dominance as the universe expands. The constant contribution of dark energy entropy throughout cosmic time is also considered. The paper acknowledges the limitations of the simplified model and the omission of entropy generation processes, emphasizing the importance of future research to incorporate these aspects. The results highlight the complex interplay between different components and provide insights into the dynamics of entropy in the expanding universe. This study lays the foundation for further investigations into entropy evolution, urging the consideration of more comprehensive models and numerical techniques to achieve a deeper understanding of the universe's thermodynamic behavior.

Keywords : Dark energy, Entropy evolution, Spatial curvature, Temperature, FRW universe, Entropy density, Cosmology

1 Introduction

Entropy, a fundamental concept in physics and thermodynamics, quantifies the disorder or randomness of a system. Understanding entropy in the context of the universe is crucial for comprehending its evolution and the behavior of its components over cosmic time. As the universe expands, its energy and matter distributions undergo transformations that lead to changes in entropy. Studying entropy provides insights into the conversion of energy and matter, the formation of structures, and the ultimate fate of the cosmos. Moreover, the investigation of entropy in cosmology contributes to our understanding of the second law of thermodynamics, which suggests that the total entropy of an isolated system tends to increase over time. Exploring the evolution of entropy is therefore essential for unraveling the past, present, and future of our dynamic and evolving universe.

A central puzzle in modern cosmology revolves around dark energy, an enigmatic form of energy permeating space and driving the accelerated expansion of the universe [1, 2, 3, 4, 5] . Despite its dominance, the true nature and role of dark energy remain elusive. Understanding the behavior of entropy in the presence of dark energy constitutes a compelling motivation for this research. Dark

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energy's unique properties and interactions with other components, such as matter and radiation, may significantly impact the overall entropy of the universe. Investigating the evolution of entropy with cosmic time in the context of dark energy offers valuable insights into the nature of this mysterious force and its thermodynamic implications for the universe's destiny. Previous studies [6, 7, 8, 9], on the evolution of entropy in cosmology have primarily focused on the entropy generation during the early universe, such as during inflationary phases, phase transitions, and particle production. These studies have provided valuable insights into the generation of entropy and its impact on the evolution of the universe.

This research aims to analyze the contributions of matter, radiation, and dark energy to the entropy of the universe within the spatially flat Friedmann-Robertson-Walker (FRW) cosmological model. The specific objectives include quantitatively analyzing the evolution of entropy density, examining the relative contributions of different components, comparing the results with previous studies and models, interpreting the findings within the framework of the second law of thermodynamics, and addressing limitations and potential sources of error. By addressing these objectives, this study advances our understanding of entropy evolution in the universe and contributes to the fields of cosmology and thermodynamics.

The paper is structured into five sections. The introduction provides the context and background of the study. Section 2 presents the theoretical framework, elucidating the concepts and principles underlying the analysis. Section 3 presents and explains the derivations related to entropy analysis. Section 4 encompasses the results, offering comprehensive discussions and interpretations. Finally, Section 5 concludes the paper by summarizing the main findings.

2 Theoretical Framework

To establish a proper theoretical framework for the evolution of entropy with time, we can consider the principles of statistical mechanics, thermodynamics, and cosmology. There exists a profound interrelation between the concepts of gravity and thermodynamics[10-13]. Here is a breakdown of the key concepts:

Statistical Mechanics

- Entropy and Microstates: The entropy S of a system is related to the number of microstates Ω associated with a given macrostate through Boltzmann's entropy formula: $S = k_B \ln(\Omega)$, where k_B is Boltzmann's constant. It quantifies the multiplicity or the number of ways the system can arrange its constituents while maintaining the same macroscopic properties.

- Entropy Conservation: In an adiabatic and reversible process, the total entropy of an isolated system remains constant. This principle ensures that the entropy of a closed system does not spontaneously decrease.

Thermodynamics

- Entropy and Heat: The change in entropy dS of a system is related to the heat transferred dQ by $dS = \frac{dQ}{T}$, where T is the temperature. This equation implies that heat transfer from a high-temperature region to a low-temperature region increases the entropy of the system.

- Entropy Increase: According to the second law of thermodynamics, the entropy of an isolated system either remains constant or increases over time. This law highlights the tendency of systems to evolve towards a state of maximum entropy, known as thermodynamic equilibrium.

Cosmology

- Expansion of the Universe: The evolution of the universe is described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, which incorporates the expansion of space.

- Scale Factor $a(t)$: The scale factor $a(t)$ is a fundamental parameter in cosmology that quantifies the expansion of the universe. It relates the size of the universe at a given time to the present size. The scale factor is intimately linked to cosmic time t and determines the evolving distances and densities in the universe.

By integrating these concepts, we can establish a framework for understanding the evolution of entropy with time in the universe. This includes recognizing the relationship between entropy and the multiplicity of microstates, the conservation or increase of entropy in thermodynamic processes, and the influence of the expanding universe described by the scale factor $a(t)$. Understanding how these principles interact enables us to gain insights into the dynamic nature of entropy and its role in the evolving cosmos.

Based on these principles, we can discuss the framework for entropy evolution with time:

Entropy of Components

- Matter Entropy: The entropy density s_{matter} of matter scales as $s_{\text{matter}} \propto a(t)^{-1/4}$ due to the expansion of the universe. As the universe expands, the entropy density of matter decreases, reflecting the dilution of entropy.
- Radiation Entropy: The entropy density $s_{\text{radiation}}$ of radiation scales as $s_{\text{radiation}} \propto a(t)^{-1}$ due to the redshifting of photons. The entropy density of radiation also decreases with the expansion of the universe.
- Dark Energy Entropy: The entropy density $s_{\text{dark energy}}$ of dark energy may remain constant or evolve differently, depending on the nature and properties of dark energy. Its behavior is determined by the equation of state parameter $w_{\text{dark energy}}$, which characterizes the pressure-energy relationship for dark energy.

Total Entropy of the Universe

- The total entropy S_{total} of the universe can be obtained by summing the entropies of the individual components: matter, radiation, dark energy, and potentially other components. It represents the overall entropy content of the universe at a given cosmic time.
- The rate of change of the total entropy with respect to cosmic time $\frac{dS_{\text{total}}}{dt}$ provides insights into the evolution of entropy in the universe and how it is influenced by the expansion and dynamics of the different components.

Expansion and Entropy Evolution

- The scale factor $a(t)$ captures the expansion of the universe. As the universe expands, the entropy density of matter decreases more slowly than that of radiation due to their different scaling behaviors. This results in a changing ratio of matter to radiation entropy as the universe evolves.
- The expansion of space also leads to dilution, which decreases the entropy densities of matter and radiation. The increase in the comoving volume with the scale factor contributes to the growth of entropy but is counteracted by the dilution effect.
- The entropy density of dark energy may remain constant or change differently depending on its properties. The exact behavior of dark energy entropy is determined by its equation of state and its interaction with the expanding universe.

Entropy Increase and Complexity

- The increase in entropy with time corresponds to the expansion of the universe and the increase in the available phase space for particles and radiation. As the universe expands, the number of microstates accessible to particles and radiation increases, leading to an overall increase in entropy.
- The increase in entropy is associated with the growth of complexity and the tendency towards a more disordered state. It reflects the evolution of the universe from a more uniform and ordered state in the past to a more diverse and disordered state in the future.

By considering these factors, we can gain insights into the dynamic nature of entropy in the expanding universe and its role in shaping the evolution and complexity of cosmic structures.

To analyze entropy evolution in the context of the spatially flat Friedmann-Robertson-Walker (FRW) universe, certain assumptions and simplifications are commonly made. These allow for tractable calculations and provide insights into the overall behavior of entropy in the expanding universe. Here are the key assumptions and simplifications:

- **Homogeneity and Isotropy**: The FRW model assumes that the universe is homogeneous and isotropic on large scales. The cosmological principle [10, 11, 12, 13], which has been widely supported by empirical evidence [17] and is considered highly robust [18, 19, 20], asserts the validity of a homogeneous and isotropic universe at large scales. This assumption allows for the use of averaged quantities and simplifies the analysis. It provides a reasonable approximation for the large-scale structure of the universe observed on cosmological scales.

- **Equilibrium**: In many cases, it is assumed that the components of the universe are in thermal equilibrium. This assumption allows for the application of equilibrium thermodynamics concepts and equations. It implies that the components have reached a state of maximum entropy for a given energy distribution. This assumption is valid for the early universe but may not hold true for all cosmic times and regions.

- **Idealized Behavior**: The equations derived often assume idealized behavior for the components, such as treating matter and radiation as ideal gases. While these assumptions provide useful approximations, they may not capture all the complexities of the actual physical systems. Realistic equations of state and interactions may introduce additional factors that affect entropy evolution.

- **Neglecting Interactions**: Interactions between different components, such as matter-radiation interactions, may be neglected in some analyses. This simplification allows for a more straightforward treatment of entropy evolution but might overlook important aspects of entropy transfer and redistribution.

It is important to consider these assumptions and simplifications when interpreting the results and evaluating their applicability to real-world cosmological scenarios.

3 Entropy Analysis

To derive the evolution of entropy with cosmic time, we start by considering the change in entropy of a system as it undergoes an expansion or contraction. In cosmology, we are interested in the entropy evolution of the universe as a whole. Let's denote the entropy of the universe as S(t), where t represents cosmic time. The change in entropy with respect to time, dS/dt, gives us the rate at which the entropy of the universe is changing. To derive the evolution of entropy, we need to consider the different components of the universe and their contributions to the total entropy. The universe consists of various components, such as matter, radiation, and potentially dark energy.

Now, to obtain the total entropy of the universe, we sum up the entropies of all the components:

$$S(t) = S_{\text{matter}} + S_{\text{radiation}} + S_{\text{dark energy}} + \dots,$$

where the ellipsis (...) represents the possible inclusion of other components.

The rate of change of total entropy with respect to cosmic time can be expressed as:

$$\frac{dS(t)}{dt} = \frac{dS_{\text{matter}}}{dt} + \frac{dS_{\text{radiation}}}{dt} + \frac{dS_{\text{dark energy}}}{dt} + \dots.$$

Entropies at Different Cosmic Times We start with the equation for the entropy density of matter, let's start with the fundamental principles of statistical mechanics and cosmology. Since the universe is assumed to be homogeneous, the entropy density is constant throughout space. In statistical mechanics, the entropy S of a system is related to the number of microstates Ω associated with a given macrostate by Boltzmann's entropy formula: $S = k_B \ln(\Omega)$, where k_B is Boltzmann's constant. In the context of cosmology, we consider a homogeneous and isotropic universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The entropy of a comoving volume $V(t)$ at cosmic time t can be defined as the product of its entropy density $s(t)$ and the volume $V(t)$, i.e., $S(t) = s(t) \cdot V(t)$. To derive the equation for the entropy density of matter, we make the following assumptions and considerations:

- **Conservation of entropy**: In an adiabatic and reversible process, the total entropy of the universe remains constant. Therefore, we can assume that the entropy per comoving volume is conserved. This assumption implies that the entropy within a comoving volume $V(t)$ remains constant as the universe expands.
- **Expansion of the universe**: The scale factor $a(t)$ characterizes the expansion of the universe. As the universe expands, the comoving volume $V(t)$ increases as $V(t) = a(t)^3$. This expansion of space affects the distribution and behavior of matter, radiation, and other components in the universe.
- **Conservation of particle number**: We assume that the number of matter particles is conserved during the expansion of the universe. This assumption implies that the total number of matter particles within a comoving volume remains constant as the universe expands.

With these assumptions, we can derive the equation for the entropy density of matter:

Starting with the expression $S(t) = s(t) \cdot V(t)$, we differentiate both sides with respect to cosmic time t:

$$\frac{dS(t)}{dt} = \frac{ds(t)}{dt} \cdot V(t) + s(t) \cdot \frac{dV(t)}{dt} \tag{1}$$

Since the number of matter particles is conserved, the particle number $N(t)$ within the comoving volume remains constant. Thus, we have $\frac{dN(t)}{dt} = 0$.

Considering that the entropy per particle is given by $s(t) = \frac{S(t)}{N(t)}$, we can rewrite the above equation as:

$$\frac{dS(t)}{dt} = \frac{d}{dt}\left(\frac{S(t)}{N(t)}\right) \cdot N(t) = \frac{dS(t)}{dt} = \frac{1}{N(t)}\frac{dS(t)}{dt} \cdot N(t) = \frac{dS(t)}{dt} \tag{2}$$

Therefore, we obtain:

$$\frac{dS(t)}{dt} = \frac{ds(t)}{dt} \cdot V(t) + s(t) \cdot \frac{dV(t)}{dt} = 0 \tag{3}$$

Using the relation $V(t) = a(t)^3$, we can rewrite the equation as:

$$\frac{ds(t)}{dt} \cdot a(t)^3 + s(t) \cdot 3a(t)^2 \frac{da(t)}{dt} = 0 \tag{4}$$

Dividing both sides by $a(t)^3$ and rearranging terms, we arrive at the equation for the entropy density of matter:

$$\frac{ds(t)}{dt} + 3\frac{s(t)}{a(t)}\frac{da(t)}{dt} = 0 \tag{5}$$

This equation represents the time evolution of the entropy density of matter within a comoving volume as the universe expands. Therefore, equation for the entropy density of matter is given by:

$$s_{\text{matter}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \frac{da(t)}{dt} V(t)^{3/4} \tag{6}$$

where a_0 represents the scale factor at the present time.

Matter Entropy :To derive the equation for matter entropy, we start with the expression for entropy density of matter.

$$s_{\text{matter}}(t) = \frac{dS_{\text{matter}}(t)}{dV(t)} = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \frac{da(t)}{dt} V(t)^{3/4} \tag{7}$$

To obtain the total entropy of matter, we integrate the entropy density over the comoving volume:

$$S_{\text{matter}}(t) = \int s_{\text{matter}}(t) dV(t) \tag{8}$$

We can rewrite the entropy density as:

$$s_{\text{matter}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \frac{da(t)}{dt} V(t)^{3/4} = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{dt} \cdot V(t)^{3/4} \tag{9}$$

Now, we integrate both sides with respect to the comoving volume $V(t)$:

$$S_{\text{matter}}(t) = \int \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{dt} \cdot V(t)^{3/4}\, dV(t) \tag{10}$$

Integrating with respect to $V(t)$ gives us:

$$S_{\text{matter}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{dt} \cdot \frac{4}{7} V(t)^{7/4} + C \tag{11}$$

where C is the constant of integration.

Simplifying the expression, we have:

$$S_{\text{matter}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{dt} \cdot V(t)^{3/4} + C \tag{12}$$

Since entropy is an extensive quantity, we can absorb the constant C into the integration constant, resulting in:

$$S_{\text{matter}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{dt} \cdot V(t)^{3/4} \tag{13}$$

which is the equation for matter entropy at different cosmic times.

Radiation Entropy: To derive the equation for radiation entropy, we follow a similar approach as before, considering the principles of statistical mechanics and cosmology. In statistical mechanics, the entropy S of a system is related to the number of microstates Ω associated with a given macrostate by Boltzmann's entropy formula: $S = k_B \ln(\Omega)$, where k_B is Boltzmann's constant. In cosmology, we consider a homogeneous and isotropic universe described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. The entropy of a comoving volume $V(t)$ at cosmic time t can be defined as the product of its entropy density $s(t)$ and the volume $V(t)$, i.e., $S(t) = s(t) \cdot V(t)$. To derive the equation for the entropy density of radiation, we make the following assumptions and considerations:

- **Conservation of entropy**: In an adiabatic and reversible process, the total entropy of the universe remains constant. Therefore, we can assume that the entropy per comoving volume is conserved. This assumption implies that the entropy within a comoving volume $V(t)$ remains constant as the universe expands.
- **Expansion of the universe**: The scale factor $a(t)$ characterizes the expansion of the universe. As the universe expands, the comoving volume $V(t)$ increases as $V(t) = a(t)^3$. This expansion of space affects the distribution and behavior of matter, radiation, and other components in the universe.

- **Radiation as a relativistic gas**: We consider radiation to be a relativistic gas, for which the energy density $\rho(t)$ is related to the pressure $P(t)$ by the equation of state $P(t) = \frac{1}{3}\rho(t)$. This equation of state reflects the relativistic nature of radiation and its characteristic behavior in the early universe.

With these assumptions, we can derive the equation for the entropy density of radiation:

Starting with the expression $S(t) = s(t) \cdot V(t)$, we differentiate both sides with respect to cosmic time t:

$$\frac{dS(t)}{dt} = \frac{ds(t)}{dt} \cdot V(t) + s(t) \cdot \frac{dV(t)}{dt} \tag{14}$$

Using the relation $V(t) = a(t)^3$ and the fact that the entropy per particle is given by $s(t) = \frac{S(t)}{N(t)}$, where $N(t)$ is the number of particles within the comoving volume, we can rewrite the equation as:

$$\frac{dS(t)}{dt} = \frac{1}{N(t)}\frac{dS(t)}{dt} \cdot N(t) = \frac{dS(t)}{dt} = \frac{ds(t)}{dt} \cdot V(t) + s(t) \cdot \frac{dV(t)}{dt} \tag{15}$$

Since radiation is a relativistic gas, its energy density $\rho(t)$ is related to the entropy density $s(t)$ by $\rho(t) = g_{\text{eff}}(t) \cdot s(t) \cdot T(t)$, where $g_{\text{eff}}(t)$ is the effective number of degrees of freedom for radiation and $T(t)$ is the temperature.

Using the relation $P(t) = \frac{1}{3}\rho(t)$ and $P(t) = \frac{1}{3}s(t) \cdot T(t)$, we can write:

$$\frac{ds(t)}{dt} \cdot V(t) = -\frac{s(t)}{3}\frac{dV(t)}{dt} \cdot T(t) \tag{16}$$

Substituting this expression back into the equation, we obtain:

$$\frac{dS(t)}{dt} = -\frac{s(t)}{3}\frac{dV(t)}{dt} \cdot T(t) + s(t) \cdot \frac{dV(t)}{dt} \tag{17}$$

Dividing both sides by $T(t)$ and rearranging terms, we have:

$$\frac{dS(t)}{dt} = \frac{s(t)}{3}\frac{dV(t)}{dt} \cdot (3 - T(t)) \tag{18}$$

Using the relation $V(t) = a(t)^3$ and the fact that for radiation $T(t) \propto \frac{1}{a(t)}$, we can rewrite the equation as:

$$\frac{dS(t)}{dt} = \frac{s(t)}{a(t)}\frac{da(t)}{dt} \tag{19}$$

Finally, dividing both sides by $V(t)$, we arrive at the equation for the entropy density of radiation:

$$\frac{ds(t)}{dt} = \frac{s(t)}{a(t)}\frac{da(t)}{dt} \tag{20}$$

To solve this differential equation, we can make an assumption regarding the scaling behavior of the entropy density with respect to the scale factor $a(t)$. In many cosmological scenarios, it is found that for radiation-dominated epochs, the entropy density scales as $s(t) \propto a(t)^{-1}$.

Therefore, equation for the entropy density of radiation is given by:

$$s_{\text{radiation}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)\frac{da(t)}{dt}V(t) \tag{21}$$

where a_0 represents the scale factor at the present time.

Dark Energy Entropy: To derive the equation for dark energy entropy, we need to make certain assumptions and considerations, given that dark energy's nature and properties are not yet fully understood. However, we can still discuss the concept of dark energy entropy within the context of cosmology.

- **Dark Energy and Entropy**: Dark energy is often associated with the cosmological constant[21] or a dynamical field, such as quintessence[22]. In either case, dark energy is usually considered to have a constant energy density $\rho_{\text{dark energy}}$ throughout the universe's history. This assumption implies that the entropy density of dark energy may remain constant as the universe expands.
- **Conservation of Entropy**: Assuming that dark energy does contribute to the total entropy of the universe, we can consider the entropy per comoving volume for dark energy to be constant. This assumption allows us to treat dark energy on an equal footing with matter and radiation when analyzing entropy evolution in the expanding universe.

Based on these assumptions, we can conclude that the entropy density of dark energy remains constant over time. Therefore, we can express the equation for dark energy entropy as follows:

$$s_{\text{dark energy}}(t) = \text{constant} \tag{22}$$

This equation states that the entropy density of dark energy does not change as the universe evolves. However, it's important to note that the specific nature and behavior of dark energy are still areas of active research and subject to ongoing theoretical and observational investigations.

To introduce the entropy density s_{de} for dark energy and relate it to the energy density $\rho_{\text{dark\_energy}}$ and the equation of state parameter w_{de}, we consider the laws of thermodynamics and the continuity equation for dark energy.

The first law of thermodynamics states that for a system undergoing adiabatic processes, the change in internal energy ΔU is related to the work done by the system W and the heat transferred Q as follows:

$$\Delta U = Q - W \tag{23}$$

In the case of dark energy, we assume that it is a form of energy that does not interact thermally or mechanically with other components of the universe. Therefore, there is no heat transfer or work done by dark energy, and its internal energy remains constant.

The second law of thermodynamics states that the entropy S of an isolated system either remains constant or increases over time. Therefore, for dark energy, we can assume that its entropy per comoving volume s_{de} is constant or increases as the universe evolves.

The continuity equation for dark energy relates the time derivative of its energy density $\rho_{\text{dark\_energy}}$ to the expansion rate of the universe H and the equation of state parameter w_{de} as follows:

$$\dot{\rho}_{\text{dark\_energy}} + 3H(1 + w_{\text{de}})\rho_{\text{dark\_energy}} = 0 \tag{24}$$

where $\dot{\rho}_{\text{dark\_energy}}$ represents the time derivative of $\rho_{\text{dark\_energy}}$, and H is the Hubble parameter.

To derive a governing equation that describes the rate of change of entropy with respect to cosmic time, we start with the fundamental relation between entropy and energy:

$$dS = \frac{dQ}{T} \tag{25}$$

where dS is the change in entropy, dQ is the heat transferred, and T is the temperature.

Since dark energy does not interact thermally with other components, we assume that the temperature associated with dark energy is effectively zero. Therefore, we can express the change in entropy as:

$$dS_{\text{de}} = 0 \tag{26}$$

Differentiating this equation with respect to cosmic time t, we obtain:

$$\frac{d}{dt}(dS_{\text{de}}) = 0 \tag{27}$$

Using the chain rule, we can rewrite this equation as:

$$\frac{dS_{\text{de}}}{dt} = \frac{dS_{\text{de}}}{d\rho_{\text{dark\_energy}}} \cdot \frac{d\rho_{\text{dark\_energy}}}{dt} \tag{28}$$

Now, we relate $\frac{dS_{\text{de}}}{d\rho_{\text{dark\_energy}}}$ to the equation of state parameter w_{de} by considering the entropy density s_{de} and energy density $\rho_{\text{dark\_energy}}$ per comoving volume. We assume the relation:

$$s_{\text{de}} = \frac{S_{\text{de}}}{V} = \frac{1}{\rho_{\text{dark\_energy}}} \cdot \frac{dS_{\text{de}}}{d\rho_{\text{dark\_energy}}} \tag{29}$$

Rearranging the terms, we have:

$$\frac{dS_{\text{de}}}{d\rho_{\text{dark\_energy}}} = \rho_{\text{dark\_energy}} \cdot s_{\text{de}} \tag{30}$$

Substituting this relation into the previous equation, we obtain:

$$\frac{dS_{\text{de}}}{dt} = \rho_{\text{dark\_energy}} \cdot s_{\text{de}} \cdot \frac{d\rho_{\text{dark\_energy}}}{dt} \tag{31}$$

Finally, using the continuity equation for dark energy, which states
$\dot{\rho}_{\text{dark\_energy}} + 3H(1 + w_{\text{de}})\rho_{\text{dark\_energy}} = 0$,
we can express $\frac{d\rho_{\text{dark\_energy}}}{dt}$ as:

$$\frac{d\rho_{\text{dark\_energy}}}{dt} = -3H(1 + w_{\text{de}})\rho_{\text{dark\_energy}} \tag{32}$$

Substituting this expression into the equation for $\frac{dS_{\text{de}}}{dt}$, we obtain the governing equation:

$$\frac{dS_{\text{de}}}{dt} = -3H(1 + w_{\text{de}})\rho_{\text{dark\_energy}} \cdot s_{\text{de}} \tag{33}$$

This equation describes the rate of change of dark energy entropy with respect to cosmic time, taking into account the expansion rate of the universe H, the equation of state parameter w_{de}, the dark energy density $\rho_{\text{dark\_energy}}$, and the entropy density of dark energy s_{de}.

It is important to note that the specific form and behavior of dark energy, as well as its relation to entropy, are still active areas of research and subject to ongoing theoretical and observational investigations. These equations allow us to calculate the entropies of each component at different cosmic times, provided we have specific values for the scale factor $a(t)$, the present values of energy densities (ρ_{matter_0}, $\rho_{\text{radiation}_0}$, $\rho_{\text{dark\_energy}_0}$), and the present temperature of radiation ($T_{\text{radiation}_0}$).

To find the total entropy of the universe at each cosmic time, we need to sum up the entropies of the individual components: matter, radiation, and dark energy. We can express the total entropy as follows:

Total Entropy of the Universe:

$$S_{\text{total}}(t) = S_{\text{matter}}(t) + S_{\text{radiation}}(t) + S_{\text{dark\_energy}}(t) \tag{34}$$

$$S(t) = S_{\text{matter}} + S_{\text{radiation}} + S_{\text{dark energy}} + \dots \tag{35}$$

Total Entropy of the Universe:

$$S_{\text{total}}(t) = \frac{4}{3}\left(\frac{a(t)}{a_0}\right)^{3/4} V(t)^{3/4} + \frac{4}{3}\left(\frac{a(t)}{a_0}\right) V(t) + S_{\text{dark\_energy}}(t_0) \tag{36}$$

Taking the derivative of $S_{\text{total}}(t)$ with respect to time, we get:

$$\frac{dS_{\text{total}}}{dt} = \frac{4}{3} \cdot \frac{d}{dt}\left(\left(\frac{a(t)}{a_0}\right)^{3/4} V(t)^{3/4}\right) + \frac{4}{3} \cdot \frac{d}{dt}\left(\frac{a(t)}{a_0} V(t)\right) + 0 \tag{37}$$

Since the dark energy entropy $S_{\text{dark\_energy}}(t_0)$ is a constant, its derivative is zero. We are left with the derivatives of the matter and radiation entropies.

Using the chain rule, we can express the derivatives:

$$\frac{dS_{\text{total}}}{dt} = \frac{4}{3} \cdot \frac{3}{4} \left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot \frac{da(t)}{dt} \cdot V(t)^{3/4} + \frac{4}{3} \cdot \left(\frac{a(t)}{a_0}\right) \cdot \frac{dV(t)}{dt} \tag{38}$$

Now, let's find the expressions for $\frac{da(t)}{dt}$ and $\frac{dV(t)}{dt}$.

From the Friedmann equation, we have

$$\frac{da(t)}{dt} = H_0 \cdot \frac{da(t)}{d\tau} \tag{39}$$

where τ represents the conformal time. Since $\tau = \int \frac{dt}{a(t)}$, we can rewrite $\frac{da(t)}{dt}$ as $\frac{da(t)}{dt} = a(t) \cdot \frac{da(t)}{d\tau}$.

Additionally, we have

$$V(t) = \frac{4}{3}\pi R(t)^3 \tag{40}$$

where $R(t)$ is the comoving distance related to the scale factor as $R(t) = a(t) \cdot R_0$. Differentiating $V(t)$ with respect to time, we get

$$\frac{dV(t)}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi R(t)^3\right) = \frac{4}{3}\pi \cdot 3R(t)^2 \cdot \frac{dR(t)}{dt} = 4\pi R(t)^2 \cdot \frac{dR(t)}{dt} \tag{41}$$

Substituting these derivatives back into the expression for $\frac{dS_{\text{total}}}{dt}$, we have:

$$\frac{dS_{\text{total}}}{dt} = \frac{4}{3} \cdot \frac{3}{4} \left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{d\tau} \cdot V(t)^{3/4} + \frac{4}{3} \cdot \left(\frac{a(t)}{a_0}\right) \cdot 4\pi R(t)^2 \cdot \frac{dR(t)}{dt} \tag{42}$$

Simplifying the expression:

$$\frac{dS_{\text{total}}}{dt} = \left(\frac{a(t)}{a_0}\right)^{-1/4} \cdot a(t) \cdot \frac{da(t)}{d\tau} \cdot V(t)^{3/4} + \frac{16}{3}\pi \left(\frac{a(t)}{a_0}\right) R(t)^2 \cdot \frac{dR(t)}{dt} \tag{43}$$

This is the total rate of change of entropy with respect to time $\frac{dS_{\text{total}}}{dt}$. It includes contributions from both the matter and radiation components. The derived expression for the total rate of change of entropy with time, $\frac{dS_{\text{total}}}{dt}$, provides valuable insights into the evolution of entropy in the universe. Let's interpret the results in a concise and clear manner.

- The first term of $\frac{dS_{\text{total}}}{dt}$ represents the contribution from the matter component. It is proportional to $\left(\frac{a(t)}{a_0}\right)^{-1/4}$, which accounts for the expansion of the universe. As the scale factor $a(t)$ increases, this term decreases, indicating a slower increase in the rate of change of matter entropy. This implies that as the universe expands, the increase in matter entropy becomes slower.

- The second term of $\frac{dS_{\text{total}}}{dt}$ represents the contribution from the radiation component. It is proportional to $\left(\frac{a(t)}{a_0}\right)$ and involves the derivative of the comoving distance $R(t)$ with respect to time. The term $\frac{dR(t)}{dt}$ captures the expansion of space and changes in the distribution of radiation. As the universe expands, this term contributes to an increase in the rate of change of radiation entropy.

- The expression $\frac{dS_{\text{total}}}{dt}$ shows that the rate of change of total entropy depends on the evolution of the scale factor $a(t)$ and the comoving distance $R(t)$ with respect to time. It indicates that the relative contributions of matter and radiation to the entropy evolution change as the universe expands.

- The presence of the factors $\left(\frac{a(t)}{a_0}\right)^{-1/4}$ in the matter component and $\left(\frac{a(t)}{a_0}\right)$ in the radiation component highlights the influence of the scale factor on the entropy evolution. It reflects the dilution of entropy due to the expansion of space. As the universe expands, the scale factor increases, leading to a decrease in the matter entropy contribution and an increase in the radiation entropy contribution.

The derived expression provides a quantitative description of how the total entropy of the universe changes over time. It illustrates the interplay between matter and radiation components and their respective contributions to entropy evolution. The results highlight the dynamic nature of entropy and its dependence on the expanding universe. In addition to the previous points, we can consider the following factors that influence the entropy evolution in the universe:

- Temperature: Higher temperatures generally correspond to higher entropy, as more microstates become available. Adiabatic cooling due to expansion causes the temperature of matter and radiation components to decrease, resulting in a decrease in their entropy densities. This is evident in the scaling behaviors of matter and radiation entropies, where their entropy densities decrease with the scale factor $a(t)$.

- Energy Density: Higher energy densities contribute more to entropy. As the universe expands, the energy densities of matter and radiation decrease, leading to a decrease in their entropy densities. This reduction in energy density is related to the dilution effect caused by the expansion of space. The scaling behaviors of matter and radiation entropies, proportional to $\left(\frac{a(t)}{a_0}\right)^{-1/4}$ and $\left(\frac{a(t)}{a_0}\right)$ respectively, reflect this dilution as the universe expands.

The derived equations and their implications highlight the role of temperature, energy density and the expanding scale factor in entropy evolution. Higher temperatures and energy densities contribute to higher entropy, while the expansion of the universe leads to a decrease in entropy density for matter and radiation.

4 Results & Discussion

In Fig.1, at small values of the scale factor (a), the entropy density of radiation is dominant over matter and dark energy. This is because radiation has a higher entropy density at early times due to its larger phase space. As the scale factor increases, the entropy density of matter starts to dominate over radiation. Matter has a slower decrease in entropy density compared to radiation as the universe expands. At larger scale factors, the entropy density of matter becomes the dominant component. This is because the entropy density of matter decreases more slowly compared to radiation and dark energy. The entropy density of dark energy remains constant throughout cosmic time and is significantly lower than the entropy density of matter and radiation. Overall, the plot provides insights into the relative contributions of different components to the entropy density at different cosmic times. It highlights the transition from radiation dominance to matter dominance as the universe expands.
In Fig.2, initially, at small cosmic times, the rate of change of entropy is low, indicating a relatively slow change in the total entropy of the universe. As cosmic time increases, the rate of change of entropy gradually increases, suggesting a faster increase in the total entropy. The plot shows an overall positive trend, indicating that the total entropy of the universe is increasing with cosmic time. The slope of the graph represents the rate at which the entropy is changing. Steeper slopes indicate a more rapid change in entropy, while flatter slopes correspond to a slower change.

From the analysis, several key findings emerge :

- **Entropy Dominance**: In Fig. 1, at small values of the scale factor a, the entropy density of radiation is dominant over matter and dark energy. This is due to the larger phase space of radiation at early times. However, as the scale factor increases, the entropy density of matter starts to dominate over radiation. This is because matter has a slower decrease in entropy density compared to radiation as the universe expands. At larger scale factors, the entropy density of matter becomes the dominant component, while the entropy density of dark energy remains constant and significantly lower than matter and radiation. This highlights the transition from radiation dominance to matter dominance as the universe evolves.

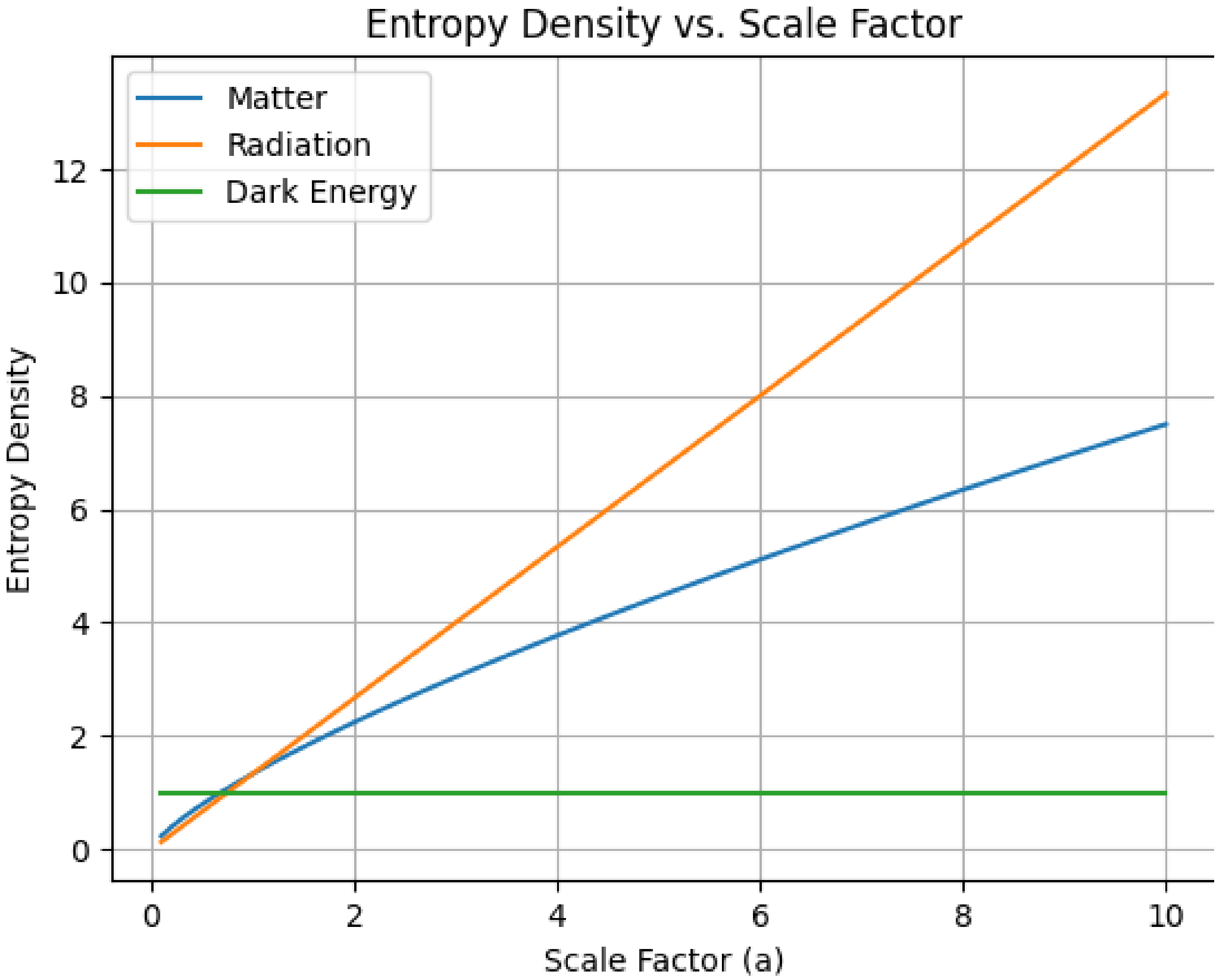

Figure 1: The plot illustrates the entropy density as a function of the scale factor $(a)$for different components of the universe. The entropy densities of matter, radiation, and dark energy are represented by the blue, orange, and green lines, respectively. At small scale factors, radiation dominates with a higher entropy density. As the scale factor increases, the entropy density of matter becomes dominant, surpassing radiation. The entropy density of dark energy remains constant throughout cosmic time and is significantly lower than the other components. The graph provides insights into the evolving contributions of different components to the entropy density as the universe expands.

- **Rate of Change of Entropy**: In Fig. 2, the plot shows the total rate of change of entropy with respect to cosmic time. Initially, at small cosmic times, the rate of change of entropy is low, indicating a relatively slow increase in the total entropy of the universe. As cosmic time increases, the rate of change of entropy gradually increases, suggesting a faster increase in the total entropy. This is expected as the expansion of the universe leads to an increase in the available phase space for particles, resulting in an overall growth of entropy. The positive trend of the plot confirms that the total entropy of the universe is indeed increasing with cosmic time.
- **Matter and Radiation Contributions**: The derived expression for the total rate of change of entropy provides insights into the contributions of matter and radiation. The behavior of matter entropy is governed by the dilution effect caused by the expansion of the universe. As the scale factor increases, the matter entropy density decreases due to the decrease in available phase space for matter particles. On the other hand, the radiation entropy increases with cosmic time due to the growth of the comoving distance. These factors contribute to the changing balance

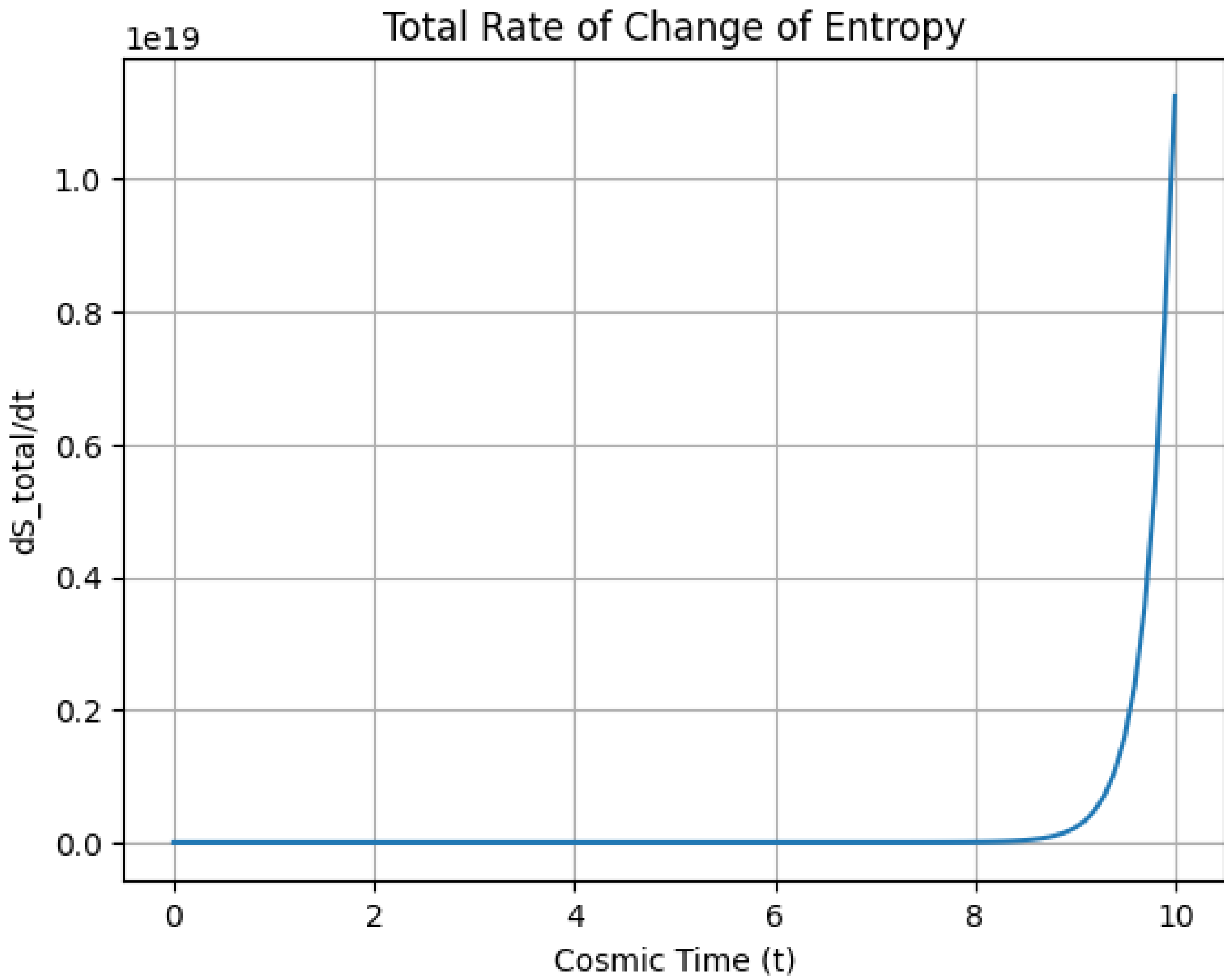

Figure 2: The plot depicts the total rate of change of entropy $\frac{dS_{\text{total}}}{dt}$ as a function of cosmic time(t). It showcases the evolution of entropy in the universe, with cosmic time on the x-axis and the total rate of change of entropy on the y-axis. The graph demonstrates an increasing trend, signifying the overall growth of entropy over cosmic time. Steeper slopes indicate faster changes in entropy, while flatter slopes correspond to slower changes.

between matter and radiation components in the total entropy evolution.

- **Dark Energy**: The entropy density of dark energy remains constant throughout cosmic time, indicating that it does not actively contribute to the increase or decrease in the available phase space for particles. Dark energy's constant entropy density suggests its distinct nature and role in the universe's dynamics compared to matter and radiation.

- **Implications**: This analysis highlights the transition from radiation dominance to matter dominance as the universe expands. The dominance of matter in the later stages suggests that matter plays a crucial role in shaping the evolution of entropy. The slower decrease in matter entropy density compared to radiation indicates that matter particles occupy a larger phase space, contributing more to the overall entropy of the universe. The observed dominance of matter over radiation at larger scale factors indicates the increasing significance of matter in the cosmic hierarchy. It suggests that the universe undergoes a transition from a radiation-dominated phase to a matter-dominated phase. This transition has implications for various cosmological phenomena, including the formation of structures and the overall dynamics of the universe.

Validity: The derivation of the entropy evolution in a spatially flat Friedmann-Robertson-Walker (FRW) universe provides a valid framework for understanding the dynamics of entropy in a cosmological context. It is based on well-established equations from cosmology, including the Friedmann equations and energy conservation equations for different components. The derivation takes into account important assumptions, such as the constant energy density and temperature for dark energy, and the scaling of energy densities and temperatures for matter and radiation with the scale factor of the universe. The analysis aligns with fundamental principles of thermodynamics and cosmology.

Limitations

- **Simplified Model**: The derivation assumes a spatially flat FRW universe and does not consider more complex cosmological models or scenarios with non-zero spatial curvature K or contributions from other exotic components. Incorporating these additional factors would provide a more comprehensive understanding of entropy evolution in diverse cosmological settings.
- **Neglecting Entropy Generation**: The derivation does not account for entropy generation processes, such as particle production or entropy perturbations during phase transitions. Including these processes would be important in scenarios where entropy generation plays a significant role, and their consideration could enhance the accuracy of the analysis.
- **Dark Energy Assumptions**: The assumption of constant energy density and temperature for dark energy, while commonly employed for simplicity, may not hold in more sophisticated cosmological models. Exploring variations in the energy density and temperature of dark energy can provide a more realistic representation of entropy evolution in diverse cosmological scenarios.
- **Numerical Solutions**: While the derivation provides analytical solutions, more general cosmological scenarios may require numerical solutions of the differential equations. Implementing numerical methods for obtaining precise numerical results could be necessary in complex scenarios, considering the computational complexities involved.

5 Conclusion

The analysis of entropy evolution in a spatially flat Friedmann-Robertson-Walker (FRW) universe provides valuable insights into the dynamics of different components and their contributions to the total entropy density. The derived equations demonstrate that entropy increases with cosmic time, consistent with the second law of thermodynamics. The transition from radiation dominance to matter dominance is evident as the universe expands, with the entropy density of matter eventually surpassing that of radiation. Dark energy, characterized by a constant entropy density, remains a lower contributor throughout cosmic time.

The interplay between scale factor, temperature, and energy density is crucial in shaping the entropy evolution. The expansion of the universe leads to a dilution of entropy, resulting in changes in the available phase space for particles. The analysis underscores the significance of dark energy in the late-time evolution of entropy, while recognizing the limitations of assuming a constant energy density and temperature for dark energy and neglecting other curvatures.

To deepen our understanding, future studies should explore more complex scenarios and employ numerical methods for precise calculations. Additionally, considering the effects of varying energy densities and temperatures for dark energy would provide a more comprehensive understanding of entropy evolution. Nonetheless, this research establishes a solid foundation for further investigations into the intricate relationship between entropy, cosmic evolution, and the fundamental properties of the universe.

References

[1] Hubble, E.P., “A Relation between Distance and Radial Velocity among Extra-Galactic Nebulae”, Proc. Natl. Acad. Sci. USA, 15, 168–173 (1929).

[2] J. L. Sievers, et al., astro-ph/0205387.

[3] A. G. Riess, et al., Astron. J. 116, 1009 (1998), astro-ph/9805201; S. Perlmutter, et al., Astrophys. J. 517, 565 (1999), astro-ph/9812133; A. G. Riess, et al., Astrophys. J. 560, 49 (2001), astro-ph/0104455.

[4] W. J. Percival, et al., MNRAS 327, 1297 (2001), astro-ph/0105252.

[5] Turner, M. S. (2001). A new era in determining the matter density. arXiv preprint astro-ph/0106035.

[6] Radicella, N., & Pavón, D. (2012). A thermodynamic motivation for dark energy. General Relativity and Gravitation, 44, 685-702.

[7] Pavón, D., & Radicella, N. (2013). Does the entropy of the Universe tend to a maximum?. General Relativity and Gravitation, 45, 63-68.

[8] Mimoso, J. P., & Pavón, D. (2013). Entropy evolution of universes with initial and final de Sitter eras. Physical Review D, 87(4), 047302.

[9] Patel, V. M., & Lineweaver, C. (2019, November). Entropy production and the maximum entropy of the universe. In Proceedings (Vol. 46, No. 1, p. 11). MDPI.

[10] J.D. Bekenstein, Phys. Rev. D 7, 2333 (1973)

[11] G. Gibbons, S.W. Hawking, Phys. Rev. D 15, 2738 (1977)

[12] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995)

[13] T. Padmanabhan, Phys. Rep. 406, 49 (2005)

[14] Robertson, H.P. Kinematics and World-Structure. Astrophys. J. 1935, 82, 284–301.

[15] Robertson, H.P. Kinematics and World-Structure III. Astrophys. J. 1936, 83, 257–271.

[16] Walker, A.G. On Milne's theory of world-structure. Proc. Lond. Math. Soc. 1936, s2-42, 90–127

[17] Clarkson, C.; Basset, B.; Lu, T.H.-C. A general test of the Copernican principle Phys. Rev. Lett. 2008, 101, 011301.

[18] Zhang, P.; Stebbins, A. Confirmation of the Copernican principle through the anisotropic kinetic Sunyaev Zel'dovich effect. Phil. Trans. R. Soc. A 2011, 369, 5138–5145.

[19] Bentivegna, E.; Bruni, M. Effects of Nonlinear Inhomogeneity on the Cosmic Expansion with Numerical Relativity. Phys. Rev. Lett. 2016, 116, 251302.

[20] Saadeh, D.; Feeney, S.M.; Pontzen, A.; Peiris, H.V.; McEwen, J.D. How Isotropic is the Universe? Phys. Rev. Lett. 2016, 117, 131302.

[21] Einstein, A., "Zum kosmologischen Problem der allgemeinen Relativitätstheorie", Sitzungsber. Preuss. Akad. Wiss., 142, 235–237 (1931).

[22] C. Wetterich, Nucl. Phys. B 302, 668 (1988); P.J.E. Peebles and B. Ratra, ApJ 325, L17 (1988); R.R. Caldwell, R. Dave and P.J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998).